ON CERTAIN HOMOLOGICAL INVARIANTS FOR COHERENT RINGS

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Abstract. In this paper, we examine the invariants $\operatorname{silp} R$ and $\operatorname{spli} R$ that were defined for any ring R by Gedrich and Gruenberg [9]. If the ring R is coherent on both sides, we show that $\operatorname{silp} R + \operatorname{silp} R^{op} = \operatorname{spli} R + \operatorname{spli} R^{op}$. In particular, if R is a commutative coherent ring, then we have an equality $\operatorname{silp} R = \operatorname{spli} R$. Our main technical tool is the decomposition of modules in the left orthogonal of certain classes of modules, established recently by Saroch and Stovicek [13].

0. Introduction

The pairing in the category of modules over a ring R defined by the functor Ext^1 induces an orthogonality relation therein, which is known to encode many properties of modules and classes of modules [8]. An important advance in the study of that orthogonality relation has been recently achieved by Saroch and Stovicek [13], who proved that if $\mathcal{B}$ is a class of modules that is closed under direct limits and direct products, then any module in the left orthogonal class ${}^{\perp}\mathcal{B}$ can be expressed as the direct limit of countably presented modules from ${}^{\perp}\mathcal{B}$. In this paper, we elaborate on that result, in order to prove the equality between two homological invariants associated with the ring R (under certain coherence assumptions).

To be more precise, we consider the invariants $\operatorname{silp} R$ and $\operatorname{spli} R$, which are defined by Gedrich and Gruenberg in [9] as the suprema of the injective lengths of projective modules and the projective lengths of injective modules, respectively. It is easy to show that these invariants are equal, whenever they are both finite. It is not clear though whether the finiteness of one of these implies the finiteness of the other, i.e. whether we always have an equality $\operatorname{silp} R = \operatorname{spli} R$. In the special case where R is an Artin algebra, the equality $\operatorname{silp} R = \operatorname{spli} R$ is equivalent to the Gorenstein Symmetry Conjecture in representation theory; cf. [1, Conjecture 13], [2, §11] and [3, Chapter VII]. In view of [10], the latter conjecture is closely related to the existence of Serre duality in the homotopy category of perfect complexes over the ring [12]. In this paper, we use the result by Saroch and Stovicek mentioned above and obtain the following:

Theorem. If the ring R is both left and right coherent, then the invariants $\operatorname{silp} R$ and $\operatorname{silp} R^{op}$ are both finite if and only if the invariants $\operatorname{spli} R$ and $\operatorname{spli} R^{op}$ are both finite.

Here, we denote by R^{op} the opposite ring of R. In particular, if R is a left (and right) coherent ring which is isomorphic with R^{op}, then $\operatorname{silp} R = \operatorname{spli} R$. In the special case of a commutative Noetherian ring, this equality was proved by Jensen in [11]. We also prove that the invariants silp and spli exhibit the following left-right (almost) symmetric behaviour for coherent rings:

If R is both left and right coherent and $\operatorname{silp} R = \operatorname{spli} R < \infty$, then $\operatorname{silp} R^{op} = \operatorname{spli} R^{op}$.

Of course, the symmetric statement is also valid.

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The contents of the paper are as follows: In Section 1, we record the properties of coherent rings that are needed in the sequel and fix our notation. In Section 2, we elaborate on the result by Saroch and Stovicek, by decomposing simultaneously a module and its first syzygy module, under the presence of suitable coherence assumptions. Finally, in Section 3 we state and prove the main results of the paper on the invariants silp and spli for coherent rings.

Notations and terminology. We work over a fixed unital associative ring R and, unless otherwise specified, all modules are left R-modules. We denote by R^{op} the opposite ring of R and do not distinguish between right R-modules and left R^{op}-modules. If $\lambda(R)$ is an invariant, which is defined in terms of a certain class of left R-modules, then $\lambda(R^{op})$ is the corresponding invariant, which is defined in terms of the appropriate class of right R-modules. Besides the invariants $\operatorname{silp} R$ and $\operatorname{spli} R$ defined above, this also applies to the invariant $\operatorname{sfli} R$, which is defined as the supremum of the flat lengths of injective modules. If $\mathcal{B}$ is a class of modules, then the left orthogonal class ${}^{\perp}\mathcal{B}$ consists of those modules M, for which $\operatorname{Ext}^1_R(M,\mathcal{B}) = 0$ (i.e. for which $\operatorname{Ext}^1_R(M,B) = 0$ for any module $B \in \mathcal{B}$).

1. Preliminary notions

In this section, we collect certain basic notions and preliminary results that will be used in the sequel. These involve the concepts of coherence, $\aleph_1$-continuity of direct systems and the notion of injectivity with respect to a class of modules.

I. Coherence. The ring R is left coherent if any finitely generated left ideal of R is finitely presented. As shown by Chase in [4], this condition is equivalent to the assertion that the class $\texttt{Flat}(R^{op})$ of flat right R-modules be closed under direct products. Modules over a left coherent ring R have the following two properties:
(i) Any finitely generated submodule $M \subseteq R^{(\mathbb{N})}$ of the free module $R^{(\mathbb{N})}$ on countable many generators is finitely presented.
Proof. Such a module M is a submodule of R^n for some $n \gg 0$ and the claim is proved using induction on n. $\square$
(ii) Any countably generated submodule $M \subseteq R^{(\mathbb{N})}$ of the free module $R^{(\mathbb{N})}$ on countable many generators is countably presented.
Proof. The module M is the ascending union of a sequence of finitely generated submodules $(M_n)_n$. Since all of the M_n's are finitely presented, the existence of a short exact sequence

$$0 \longrightarrow \bigoplus_n M_n \longrightarrow \bigoplus_n M_n \longrightarrow M \longrightarrow 0$$

shows that M is countably presented. $\square$

In particular, if R is left coherent, then any countable generated left ideal of R is countably presented. If a ring R satisfies this latter condition, we say that R is left $\aleph_0$-coherent. Then, any left coherent ring is left $\aleph_0$-coherent. Using the same arguments as above, it follows that, over a left $\aleph_0$-coherent ring R, any countably generated submodule $M \subseteq R^{(\mathbb{N})}$ of the free module $R^{(\mathbb{N})}$ on countable many generators is countably presented. Therefore, any countably presented module over a left $\aleph_0$-coherent ring admits a resolution by countably generated free modules.

II. $\aleph_1$-continuous direct systems. An ordered set $(I,\leq)$ is directed if any finite subset of I has an upper bound in I. We say that the directed set $(I,\leq)$ is $\aleph_1$-complete if any countable linearly order subset of I has a supremum in I. If Λ is any set, then the ordered set of all countable subsets $\Lambda' \subseteq \Lambda$ is $\aleph_1$-complete.

A direct system of modules $(M_i)_{i\in I}$ is $\aleph_1$-continuous if the ordered set I is $\aleph_1$-complete and for any countable linearly order subset $J \subseteq I$ with supremum $j' \in I$ the module $M_{j'}$ is the direct limit of the direct system $(M_j)_{j\in J}$. If M is a module and Λ is a set, then the direct system $\left(M^{(\Lambda')}\right)_{\Lambda'}$, where Λ' runs through all countable subsets of Λ, is $\aleph_1$-continuous.

III. $\mathcal{B}$-INJECTIVITY. Let $\mathcal{B}$ be a class of modules. Following [13], we say that a linear map $f : M \longrightarrow N$ is $\mathcal{B}$-injective provided that any linear map $M \longrightarrow B$ factors through f for any module $B \in \mathcal{B}$. If f is injective and $\operatorname{coker} f \in {}^{\perp}\mathcal{B}$, then f is $\mathcal{B}$-injective. Conversely, if f is injective, $\mathcal{B}$-injective and $N \in {}^{\perp}\mathcal{B}$, then $\operatorname{coker} f \in {}^{\perp}\mathcal{B}$. The notion of $\mathcal{B}$-injectivity has the following three properties with respect to a composable pair of morphisms $f : M \longrightarrow N$ and $g : N \longrightarrow L$:
(i) If f, g are $\mathcal{B}$-injective, then the composition $g \circ f$ is $\mathcal{B}$-injective.
(ii) If the composition $g \circ f$ is $\mathcal{B}$-injective, then f is $\mathcal{B}$-injective
(iii) If f is surjective and the composition $g \circ f$ is $\mathcal{B}$-injective, then g is $\mathcal{B}$-injective.
The proof of assertions (i) and (ii) is straightforward, whereas (iii) follows since the natural transformation f^* embeds $\operatorname{Hom}_R(N, \_)$ as a subfunctor of $\operatorname{Hom}_R(M, \_)$, if f is surjective.

2. $\aleph_0$-COHERENCE AND DECOMPOSITION OF MODULES IN ${}^{\perp}\mathcal{B}$

We consider a class of modules $\mathcal{B}$, which is closed under direct limits and direct products. As shown in [13, Theorem 2.8], any module M in the left orthogonal ${}^{\perp}\mathcal{B}$ can be expressed as the direct limit of an $\aleph_1$-continuous direct system of countably presented modules $(M_i)_i$, where $M_i \in {}^{\perp}\mathcal{B}$ and the canonical colimit map $M_i \longrightarrow M$ is $\mathcal{B}$-injective for all i. In the special case of a left $\aleph_0$-coherent ring, we aim at proving the following addendum to that result.

Proposition 2.1. *Let R be a left $\aleph_0$-coherent ring and $\mathcal{B}$ a class of modules, which is closed under direct limits and direct products. We also consider a module M with $Ext^1_R(M, \mathcal{B}) = Ext^2_R(M, \mathcal{B}) = 0$. Then, M can be expressed as the direct limit of a direct system of countably presented modules $(M_i)_i$ with $Ext^1_R(M_i, \mathcal{B}) = Ext^2_R(M_i, \mathcal{B}) = 0$ for all i.*

Proof. We consider a short exact sequence

$$0 \longrightarrow N \xrightarrow{f} F \longrightarrow M \longrightarrow 0,$$

where $F = R^{(\Lambda)}$ is a free module with basis Λ. Since $\operatorname{Ext}^1_R(N, \mathcal{B}) = \operatorname{Ext}^2_R(M, \mathcal{B}) = 0$, we apply [13, Theorem 2.8] and express N as the direct limit of an $\aleph_1$-continuous direct system $(N_i)_i$, where the module N_i is countably presented, $\operatorname{Ext}^1_R(N_i, \mathcal{B}) = 0$ and the canonical colimit map $N_i \longrightarrow N$ is $\mathcal{B}$-injective for all i. We may also express the free module F as the direct limit of the $\aleph_1$-continuous direct system $(F_j)_j$ consisting of the free submodules of F with bases the various countable subsets of Λ. Invoking [13, Construction A.1], we find an $\aleph_1$-continuous direct system of linear maps $(f_k : N_k \longrightarrow F_k)_k$, where the N_k's are some of the N_i's and the F_k's are some of the F_j's, whose direct limit is the monomorphism f. Then, the direct system $(M_k)_k$, where $M_k = \operatorname{coker} f_k$ for all k, is $\aleph_1$-continuous and $\varinjlim_k M_k = M$. Of course, the module M_k is countably presented for all k.

For any index k we consider the commutative diagram with exact rows

$$\begin{array}{ccccccccc} & & N_k & \xrightarrow{f_k} & F_k & \longrightarrow & M_k & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & N & \xrightarrow{f} & F & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

where all vertical maps are the canonical colimit maps. Since $\mathrm{Ext}^1_R(M,\mathcal{B})=0$, the linear map f is $\mathcal{B}$-injective. The vertical maps $N_k \longrightarrow N$ being also $\mathcal{B}$-injective, we conclude that f_k is $\mathcal{B}$-injective for all k. We now consider the submodule $L_k = \mathrm{im}\, f_k \subseteq F_k$ and decompose f_k as the composition $N_k \xrightarrow{p_k} L_k \xrightarrow{\iota_k} F_k$, where ι_k is the inclusion map. Since p_k is surjective, the map ι_k is $\mathcal{B}$-injective. Therefore, the freeness of F_k implies that $\mathrm{Ext}^1_R(M_k,\mathcal{B})=0$. We shall prove that the set

$$\{k : f_k \text{ is injective}\} \tag{1}$$

is a cofinal subset of the index set. This will complete the proof, since for any index k in that cofinal subset we also have $\mathrm{Ext}^2_R(M_k,\mathcal{B}) = \mathrm{Ext}^1_R(N_k,\mathcal{B}) = 0$.

For any index k the submodule $T_k = \ker f_k = \ker p_k \subseteq N_k$ is contained in the kernel of the canonical colimit map $N_k \longrightarrow N$.[1] Since M_k is countably presented and the free module F_k is countably generated, the short exact sequence

$$0 \longrightarrow L_k \xrightarrow{\iota_k} F_k \longrightarrow M_k \longrightarrow 0$$

shows that the module L_k is countably generated. In view of our hypothesis on the ring R, it follows that L_k is countably presented. The module N_k being also countably presented (and hence countably generated), the short exact sequence

$$0 \longrightarrow T_k \longrightarrow N_k \xrightarrow{p_k} L_k \longrightarrow 0 \tag{2}$$

then shows that T_k is countably generated. We are interested in the $\aleph_1$-continuous direct system of short exact sequences (2). We note that the composition $T_k \hookrightarrow N_k \longrightarrow N$ is the zero map for any index k. In view of the $\aleph_1$-continuity of the direct system $(N_k)_k$, there exists an index k' with $k' \geq k$, such that the composition $T_k \hookrightarrow N_k \longrightarrow N_{k'}$ is the zero map. It follows that the canonical map $N_k \longrightarrow N_{k'}$ factors as the composition $N_k \xrightarrow{p_k} L_k \xrightarrow{s_k} N_{k'}$ for a suitable linear map s_k

$$\begin{array}{ccccccccc}
0 & \longrightarrow & T_k & \longrightarrow & N_k & \xrightarrow{p_k} & L_k & \longrightarrow & 0 \\
 & & \downarrow 0 & & \downarrow & {}^{s_k}\swarrow & \downarrow & & \\
0 & \longrightarrow & T_{k'} & \longrightarrow & N_{k'} & \xrightarrow{p_{k'}} & L_{k'} & \longrightarrow & 0
\end{array}$$

We fix an index k and define by induction an increasing sequence of indices $(k_n)_n$, as follows: For $n=0$ we let $k_0 = k$ and for any $n \geq 1$ we let $k_n = k'_{n-1}$. If $k_\infty = \sup_n k_n$, then we claim that the map $p_{k_\infty} : N_{k_\infty} \longrightarrow L_{k_\infty}$ is injective (and hence an isomorphism). Since $k_\infty \geq k_0 = k$ and $f_{k_\infty} = \iota_{k_\infty} \circ p_{k_\infty}$, this will prove that (1) is indeed a cofinal subset of the index set. Let $\xi \in N_{k_\infty}$ be an element in the kernel of p_{k_∞}. Since $N_{k_\infty} = \varinjlim_t N_{k_t}$, there exists an integer n and an element $\xi_n \in N_{k_n}$ which maps onto ξ under the canonical map $N_{k_n} \longrightarrow N_{k_\infty}$. For any integer $m \geq n$ we denote by ξ_m the image of ξ_n under the canonical map $N_{k_n} \longrightarrow N_{k_m}$. The element ξ_n is contained in the kernel of the composition $N_{k_n} \xrightarrow{p_{k_n}} L_{k_n} \longrightarrow L_{k_\infty}$. Since $L_{k_\infty} = \varinjlim_t L_{k_t}$, there exists an integer $m \geq n$, such that ξ_n is contained in the kernel of the composition $N_{k_n} \xrightarrow{p_{k_n}} L_{k_n} \longrightarrow L_{k_m}$, i.e. in the kernel of the composition $N_{k_n} \longrightarrow N_{k_m} \xrightarrow{p_{k_m}} L_{k_m}$. It follows that $\xi_m \in \ker p_{k_m}$ and hence $\xi_{m+1} = (s_{k_m} \circ p_{k_m})\xi_m = s_{k_m}(p_{k_m}\xi_m) = s_{k_m}(0) = 0$. Since ξ is the image of ξ_{m+1} under the canonical map $N_{k_{m+1}} \longrightarrow N_{k_\infty}$, we conclude that $\xi = 0$. Therefore, $\ker p_{k_\infty} = 0$ and hence p_{k_∞} is injective, as needed. □

Remark 2.2. Let R be a left $\aleph_0$-coherent ring and consider a positive integer n and a class $\mathcal{B}$

[1]In fact, the injectivity of the canonical colimit map $F_k \longrightarrow F$ shows that $T_k = \ker(N_k \longrightarrow N)$.

of modules, which is closed under direct limits and direct products. Then, the argument used in the proof of Proposition 2.1 may be also used in order to prove the following stronger result: If M is a module, such that $\mathrm{Ext}^t_R(M,\mathcal{B})=0$ for all $t=1,\ldots,n$, then M can be expressed as the direct limit of a direct system of modules $(M_i)_i$, which admit resolutions by countably generated free modules and are such that $\mathrm{Ext}^t_R(M_i,\mathcal{B})=0$ for all $t=1,\ldots,n$ and all i.

3. The invariants silp and spli for coherent rings

We shall now apply the result on the decomposition of modules established in Section 2, in the special case where the class $\mathcal{B}$ therein is the class of flat modules. Imposing suitable coherence assumptions on the ground ring R, we will thus deduce the equality between the invariants $\operatorname{spli} R$ and $\operatorname{silp} R$.

Proposition 3.1. *Let R be a ring which is both right coherent and left $\aleph_0$-coherent. If M is a module such that $Ext^1_R(M,N)=Ext^2_R(M,N)=0$ for any flat module N, then $Tor^R_1(I,M)=0$ for any injective right module I.*

Proof. Since R is right coherent, the class $\mathcal{B}=\mathtt{Flat}(R)$ of flat modules is closed under (direct limits and) direct products. Therefore, our assumption on M and Proposition 2.1 imply that M can be expressed as the direct limit of a direct system $(M_i)_i$, where all of the modules M_i admit resolutions by countably generated free modules and $\mathrm{Ext}^1_R(M_i,N)=\mathrm{Ext}^2_R(M_i,N)=0$ for all flat modules N. In particular, $\mathrm{Ext}^1_R(M_i,R)=\mathrm{Ext}^2_R(M_i,R^{(\mathbb{N})})=0$ for all i. Invoking [6, Corollary 2.5], we conclude that $\mathrm{Tor}^R_1(I,M_i)=0$ for any injective right module I and any index i. Since $M=\varinjlim_i M_i$, the continuity of the functor Tor_1 shows that $\mathrm{Tor}^R_1(I,M)=0$ for any injective right module I, as needed. □

Corollary 3.2. *Let R be a ring, which is both right coherent and left $\aleph_0$-coherent.*
(i) $\operatorname{sfli} R^{op} \leq \operatorname{silp} R$.
(ii) $\operatorname{spli} R^{op} \leq \operatorname{silp} R + \operatorname{silp} R^{op} \leq \operatorname{silp} R + \operatorname{spli} R + \operatorname{spli} R^{op}$.
(iii) If $\operatorname{silp} R = \operatorname{spli} R < \infty$, then $\operatorname{silp} R^{op} = \operatorname{spli} R^{op}$.

Proof. (i) The inequality to be proved is obvious if $\operatorname{silp} R=\infty$ and hence we may assume that $\operatorname{silp} R = n < \infty$. Then, [7, Proposition 2.1] implies that $\mathrm{id}_R N \leq n$ for any flat module N. We fix a module M and consider the n-th syzygy module $\Omega_n M$ in some projective resolution of it; then, $\mathrm{Ext}^i_R(\Omega_n M,N)=\mathrm{Ext}^{n+i}_R(M,N)=0$ for all $i\geq 1$ and any flat module N. Therefore, Proposition 3.1 implies that $\mathrm{Tor}^R_{n+1}(I,M)=\mathrm{Tor}^R_1(I,\Omega_n M)=0$ for any injective right module I. Since this is the case for any module M, we conclude that $\mathrm{fd}_{R^{op}} I \leq n$ for any injective right module I and hence $\operatorname{sfli} R^{op} \leq n$, as needed.

(ii) The first inequality in the statement follows since

$$\operatorname{spli} R^{op} \leq \operatorname{sfli} R^{op} + \operatorname{silp} R^{op} \leq \operatorname{silp} R + \operatorname{silp} R^{op}.$$

In the above chain of inequalities, the first one follows by applying [7, Proposition 2.2] to the ring R^{op}, whereas the second one is a consequence of (i) above. The second inequality in the statement follows since for any ring R we have $\operatorname{silp} R^{op} \leq \operatorname{sfli} R + \operatorname{spli} R^{op} (\leq \operatorname{spli} R + \operatorname{spli} R^{op})$; cf. [5, Corollary 5.2] for the ring R^{op}.

(iii) If both invariants $\operatorname{silp} R$ and $\operatorname{spli} R$ are finite, then the inequalities in (ii) above imply that $\operatorname{silp} R^{op} < \infty$ if and only if $\operatorname{spli} R^{op} < \infty$. □

We may obtain a left-right symmetric assertion by assuming that the ring R is both left and right coherent, as follows:

Theorem 3.3. *Let R be a ring, which is both left and right coherent. Then, the following conditions are equivalent:*

(i) The invariants $\operatorname{silp} R$ *and* $\operatorname{silp} R^{op}$ *are finite.*

(ii) The invariants $\operatorname{spli} R$ *and* $\operatorname{spli} R^{op}$ *are finite.*

If these conditions are satisfied, then $\operatorname{silp} R = \operatorname{spli} R < \infty$ *and* $\operatorname{silp} R^{op} = \operatorname{spli} R^{op} < \infty$.

Proof. (i)$\rightarrow$(ii): In view of the left-right symmetry of the assumptions, it suffices to show that $\operatorname{spli} R^{op}$ is finite. This claim follows from the first inequality in Corollary 3.2(ii).

(ii)$\rightarrow$(i): This follows from [5, Proposition 5.3], which is valid without any coherence assumptions on R. $\square$

Corollary 3.4. *Let R be a ring which is isomorphic with its opposite R^{op}. If R is left (and hence right) coherent, then* $\operatorname{silp} R = \operatorname{spli} R$. $\square$

Corollary 3.5. *If R is a commutative coherent ring, then* $\operatorname{silp} R = \operatorname{spli} R$. $\square$

Remark 3.6. In the special case where R is a commutative Noetherian ring, the equality in Corollary 3.5 was proved by Jensen in [11, 5.9].

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